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(1761), a work of unmatched inutility. And yet good biography had commenced in force with the *Biogr. Brit.* in 1747. The total absence of historical effort encouraged the learned vicar of Twickenham, George Costard, to give to his work on the globes, (1767), full of every kind of miscellaneous historical statement, the title 'History of Astronomy.' All my reading has led me to suspect that the doubts and dangers of the disputed right to the Crown, which lasted from the rising of 1715 to that of 1745, produced a paralysing effect upon the intellectual energies of the country.

On the Application of Bonuses to limit either the term of an Assurance or the number of payments to be made under it: being Excerpts from a paper read before the Actuarial Society of Edinburgh, on 5th December, 1867. By James R. Macfadyen, *of the City of Glasgow Life Assurance Company.*

THE application of bonus to the hastening of the time fixed by the original contract for the payment of a life assurance policy, and the using of it to extinguish the premiums payable during the latter years of life, are systems, which, partly from their growing favor with the public, and consequent adoption by many offices, and still more from the fact, that the subject is by no means exhausted, notwithstanding various able articles that have appeared in this *Journal,* seem to me to render apology unnecessary for again treating of it.

The papers to which I allude as having already taken up the matter, are two articles by Mr. Sprague in the sixth volume, embracing both branches of the subject, and a more recent letter by Mr. Marr, (vol. xiii. p. 246) referring to the first of these two systems alone.

Though, as we shall hereafter see, these systems of applying bonus are very closely allied in character, and they have thus been coupled in this paper, it will be more convenient in the first place to confine ourselves to that in which the bonus is applied to limit the *term* of insurance only.

The problem usually arising in this case is to find the term of the Endowment Assurance into which a given cash bonus will convert an ordinary assurance. Mr. Sprague solves this problem by making an equation at the age the change is effected between ${}^{1}b$, the cash bonus then allocated to the policyholder, which is the benefit he relinquishes, and the difference between the costs of an Endowment Assurance for the unknown term, and an ordinary assurance for the whole of life, together with the values of all the future premiums after the maturing of the Endowment Assurance, which are the benefits given in return. From this equation, *n*, the unknown term of the Endowment Assurance can be found, and by treating this result by a similar method, the age can be found at which a second cash bonus will cause the assurance to be paid, and so with any other cash bonus that may be allocated.

Mr. Marr arrives at his formula in a different manner. He founds on the well known equation ($V_{x|y}$ representing the value of a policy at age $(x+y)$ which has been opened at age x)

$$\frac{V_{x|y}}{1+a_{x+y}}+\varpi_x=\varpi_{x+y}.$$

By making ϖ_{x+y} represent the Endowment Assurance premium for the unknown term $(n-y)$, and $1+a_{x+y}$, the annuity-due for the same period, and adding to the left-hand member the further benefit ${}^1\mathrm{b}$ given up by the assured, we have an equation formed, where *n* will appear as the unknown term of a known temporary annuity, and therefore may be readily found by a table of such annuities. By turning Mr. Marr's result into the D and N values, it can of course be reduced to the same expression as that of Mr. Sprague; and if, in either of the formulæ, the equation be manipulated so as to have ${}^1\mathrm{b}$ standing alone on one side of the expression, we have, *n* being given, the means of solving the converse problem —*i.e.* given the age at which the policy is to be payable, required to find what cash bonus is necessary for the purpose. Or if instead of ${}^1\mathrm{b}$ be substituted any other form in which the bonus may be declared, say, ${}^1\mathrm{B}\frac{M_{x+y}}{D_{x+y}}$, where ${}^1\mathrm{B}$ represents the reversionary bonus, we have the means of solving either of the preceding problems with reversionary instead of cash bonuses.

There is however a fourth method of working questions of this description, which has been alluded to, though not entered upon, in the letter of Mr. Marr already referred to, and it is to this method we shall now direct our attention.

Let $(x+y)$ be the age at first division of profit, ${}^1\mathrm{b}$, ${}^2\mathrm{b}$, ${}^3\mathrm{b}$, &c. the cash bonuses declared at ages $x+y$, $x+2y$, $x+3y$, &c., y thus being the investigation period, ${}^1\mathrm{B}$, ${}^2\mathrm{B}$, ${}^3\mathrm{B}$, &c., the corresponding reversionary bonuses, in which case ${}^1\mathrm{b}={}^1\mathrm{B}\frac{M_{x+y}}{D_{x+y}}$, ${}^2\mathrm{b}={}^2\mathrm{B}\frac{M_{x+2y}}{D_{x+2y}}$, and so on. Let ϖ_x be the *pure* premium for £1 at age of entry, x; and let n be the term of the endowment assurance, or the quantity to be found.

Then, at first period of conversion, or at age $(x+y)$ the cash bonus ${}^1\mathbf{b}$ is employed to purchase an Endowment payable at the unknown age $(x+n)$.

When that time is reached by the assured, the amount of endowment then due, together with the surrender value of his original policy, is the sum that is at his credit with the Assurance Company. But, by hypothesis, he is now entitled to receive the sum assured by his policy;—

$\therefore$ the amount of endowment then due, + the surrender value of his policy, = the sum contained in that policy = 1. From this equation we can readily find n.

Should a second bonus ${}^2\mathbf{b}$ be allocated at age $(x+2y)$, then
the amount of endowment payable at the new unknown age, that
could be purchased at age $(x+2y)$, for a sum equal to ${}^2\mathbf{b}$
+ the surrender value of the endowment payable at $(x+n)$, for
which ${}^1\mathbf{b}$ had been given
+ the surrender value, at the new unknown age, of the original
policy
= 1, as before; and thus the new unknown age could be found.

Perhaps it will be more clear if the formulæ are given. Thus—

$$\text{Value of policy} + \begin{matrix}\text{value of Endowments}\\ \text{purchased by } \mathbf{b}\end{matrix} = \text{Sum assured.}$$

or
$$A_{x+n} - \varpi_x(1+a_{x+n}) + {}^1\mathbf{b}\frac{D_{x+y}}{D_{x+n}} + {}^2\mathbf{b}\frac{D_{x+2y}}{D_{x+n}} + \&\text{c.} = 1;$$

whence,
$$1 - d(1+a_{x+n}) - \varpi_x(1+a_{x+n}) + \frac{{}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \&\text{c.}}{D_{x+n}} = 1,$$

and finally,
$$D_{x+n}(1+a_{x+n}) = \frac{{}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \&\text{c.}}{d+\varpi_x} \qquad (1)^*$$

An expression from which n can be found, whether the rates of interest and mortality assumed for the various parts of the expression be the same or not.

It will be well to point out here of what parts this formula is made up, as on the face of it, it does not clearly show, and as this must be considered in making practical use of it. The D's are calculated by the table by which the endowments are sold, the d

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* In the above formulæ, the amount of Assurance is supposed to be £1. If any other amount be wished to be employed, say S, we shall have to substitute for $d+\varpi_x$, wherever it occurs, $S(d+\varpi_x)$.

and the a_{x+n} at the rates at which the policy is valued, and the ϖ_x is the pure premium originally paid.

Should the Endowments be calculated by the same Table of Interest and Mortality as the policy is valued by, (1) will become

$$N_{x+n-1}=\frac{{}^1\mathbf{b}D_{x+y}+{}^2\mathbf{b}D_{x+2y}+\&c.}{\varpi_x+d} \qquad (2)$$

This is the expression given by Mr. Sprague; and in using his formula, it ought to be remembered that this assumption is made in it.

If in (2) the bonus is assumed to be reversionary and is valued by the same table as the policy and endowments, we have

$$N_{x+n-1}=\frac{{}^1BM_{x+y}+{}^2BM_{x+2y}+\&c.}{\varpi_x+d} \qquad (3)$$

And should still further the amounts of the reversionary bonuses declared be the same

$$N_{x+n-1}=\frac{B(M_{x+y}+M_{x+2y}+M_{x+3y}+\&c.)}{\varpi_x+d} \qquad (4)$$

From one or other of these expressions can be found n, $\mathbf{b}$, or B, whatever be the combinations of interest and mortality introduced—the only supposition made in (1), from which the others are drawn, being, that at every investigation, each endowment is sold and valued by the same table.

There are various other forms in which these equations can be considered. Thus for $\frac{1}{\varpi_x+d}$ can be substituted (should ϖ_x and d be at same rate of interest) $1+a_x$; or, should the investigation be annual, and ${}^1\mathbf{b}={}^2\mathbf{b}=\&c.$, (2) becomes

$$N_{x+n-1}=\mathbf{b}(1+a_x)(N_{x+y-1}-N_{x+\theta y}).$$

If again, as is more common, ${}^1B={}^2B=\&c.$ and the investigation is annual, (3) becomes

$$N_{x+n-1}=B(1+a_x)(R_{x+y}-R_{x+(\theta+1)y}).$$

θ being the number of the investigation immediately preceding the payment of the policy.

It will be seen from this method of solving the problem, that there is really no change in the policy, and that so far as it is concerned, it is only the surrender value of it that is paid at age $(x+n)$. The assured simply enters into a new contract with the office, by sinking his bonuses in the purchase of endowments, while the policy is in the same state as it was at entry. This principle is of the greatest importance, and will be a guide through all the various intricacies that may arise in studying the question;

and as we shall find, it holds equally good, when the bonus is applied to limit the number of payments.

It will be observed that we have taken ϖ_x as the *pure* premium at age x. If we had used the *loaded* premium, the value of the policy would be less by an annuity-due of this loading; and thus the whole fund to provide profit and pay expenses beyond age $(x+n)$, would be anticipated and pressed into the calculation for the purpose of diminishing the value of the policy. Now the assurance runs off the books of the Company at age $(x+n)$, and it would be unjust to charge the policyholder for profits he will never receive, and to debit him with expenses he will never occasion; so the *pure* premiums ought to be used.

Neither in these formulæ, nor in those of Messrs. Sprague and Marr, does the policyholder get any credit for that portion of profit accruing between $(x+\theta y)$ and $(x+n)$—and this too, although during that period he is usually called on to pay a premium which is specially designed to produce a surplus to be returned. No doubt this is an injustice, and strictly speaking, the policyholder, if the Office be one declaring intermediate bonus between investigations, should have this intermediate bonus handed over to him at age $(x+n)$, along with the amount originally assured. We are inclined to think, however, that the more general practice of Offices allowing this application of bonus, is to make no return whatever to the policyholder under this head.

Another question which it may be well to consider here, is the value of these formulæ for practical computations. A glance will show that, in this respect, the finding of n will be a matter more or less tedious, according to the number of different tables of interest and mortality on which the various parts of the equation are calculated. If these be few, the formula will be correspondingly simple, and *vice versâ*. If even the most complicated of all, (1), be employed, the operation is not a very laborious one. From Mr. Chisholm's "Commutation Tables" a table of $\lambda D_{x+n}(1+a_{x+n})$ for the Carlisle Table, can be very rapidly formed, the logarithms of the factors being there given; and should it be merely wished to find what n will become by the application of a single bonus only, it will be unnecessary to find the number corresponding to this logarithm, as the form of the other member of the equation is peculiarly adapted for the use of logarithms, and it is as easy to find n from a comparison of the logarithms as from the numbers. And the same remark may be made of the other formulæ given, as Mr. Chisholm's work supplies the logarithm of N_{x+n-1}, or, as it is there termed, N_{x+n}, for the Carlisle Table.

The use of $1+a_x$ for $\dfrac{1}{\varpi_x+d}$, when it is admissible, will be an obvious advantage.*

As, after the next section of the present paper, any further researches we may make, will be equally applicable to assurances, whether limited in term or in number of premiums, we shall now proceed to investigate the formula for finding that age at which premiums may be made to cease by the application of bonuses. Mr. Sprague, in the articles before referred to, has taken up this subject also. The method he employs to find n is this:—

The benefit the Office surrenders is the payment of premiums after age $(x+n)$, and for this the policyholder gives his cash bonus ${}^1\mathbf{b}$; which produces an equation from which n can be found; and similarly, when a second bonus is allocated, the new age can be found.

There is, however, another method of finding n, somewhat analagous to that investigated already in the limiting of the term of assurance, but of which we have not observed any previous notice taken; and it is this we shall now examine.

Let P_x be the premium *actually paid*, let $(x+n)$ here mean the age *on* and after which no further payments are to be made, and let the other symbols be as previously.

Then at age $(x+n)$ the Company foregoes all future premiums, that is, gives up a benefit of $P_x(1+a_{x+n})$. To meet this, at successive declarations of profit, the policyholder pays the Office sums ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, &c., which are allowed to accumulate to meet the single payment at $(x+n)$, requisite to redeem the premiums, that is, $P_x(1+a_{x+n})$—or, in other words, as in the first form of applying bonus, the policyholder purchases with ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, &c., endowments payable at age $(x+n)$. Therefore

$$ {}^1\mathbf{b}\frac{D_{x+y}}{D_{x+n}} + {}^2\mathbf{b}\frac{D_{x+2y}}{D_{x+n}} + \ldots {}^\theta\mathbf{b}\frac{D_{x+\theta y}}{D_{x+n}} = P_x(1+a_{x+n}), $$

therefore

$$ \frac{{}^1\mathbf{b}D_{x+y} + {}^2\mathbf{b}D_{x+2y} + \ldots {}^\theta\mathbf{b}D_{x+\theta y}}{P_x} = D_{x+n}(1+a_{x+n}) \qquad (5) $$

whence n can be found.

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* Since writing the above, Mr. Ralph P. Hardy has kindly put at my service, a method he employs to find n, when one bonus only is introduced, and both policy and Endowment are valued at same rates.

In (2), divide by D_{x+y}, and we have

$$ \frac{N_{x+n-1}}{D_{x+y}} = \frac{{}^1\mathbf{b}}{d+\varpi_x} $$

and, thus, we have the unknown member of this equation, take the form of an annuity deferred n years, of which the amount is known, as also the present age of the life. Whence n can be readily found, from a Table of such annuities.

Here, as in the previous case, the D's are calculated by the table of interest and mortality at which the policyholder purchases his endowments, and the a_{x+n} is founded on the table the Office uses in estimating the worth of the future premiums.

Should these be the same, (1) becomes

$$\frac{{}^{1}\mathbf{b}D_{x+y}+{}^{2}\mathbf{b}D_{x+2y}+\ldots{}^{\theta}\mathbf{b}D_{x+\theta y}}{P_x}=N_{x+n-1} \qquad (6)$$

which is the expression of Mr. Sprague, so that we have to repeat here the caution about using his formula without taking note of its limitations; the deferred annuity employed in it, containing both the endowments and the value of future premiums at age $(x+n)$, and those assumed at the same rates. It is an assumption, however, that practically is of little moment to attend to, as these rates will almost invariably be taken as the same.

If in (6) we wish to use the reversionary bonuses, it becomes, should they also be valued by the same table as the premiums are redeemed at, and the endowments are sold by,

$$\frac{{}^{1}BM_{x+y}+{}^{2}BM_{x+2y}+\ldots{}^{\theta}BM_{x+\theta y}}{P_x}=N_{x+n-1} \qquad (7)$$

and if $y=1$, and ${}^{1}\mathbf{b}={}^{2}\mathbf{b}={}^{3}\mathbf{b}=$&c., then (6) becomes

$$\frac{\mathbf{b}}{P_x}(N_{x+y-1}-N_{x+\theta y})=N_{x+n-1}.$$

Or if $y=1$, and ${}^{1}B={}^{2}B=$&c., then (7) becomes

$$\frac{B}{P_x}(R_{x+y}-R_{x+\overline{\theta+1}y})=N_{x+n-1}.$$

From one or other of these formulæ can be found expressions for n, **b**, or B, for any possible combination of interest and mortality rates, with the sole exception of the case when any one of the Endowments is sold or valued by a different table from the others, a form which, alike from the extreme improbability of its occurrence, and from the cumbrousness of the resulting formula, we do not feel called on now to consider.

In the application of bonus to limiting the term of assurance, we used ϖ_x the pure premium; but, in this case it must be the actual premium paid, as after age $(x+n)$ the policy will still continue on the books of the Company, draw profits, and cause expense. The single item of the loading that might possibly be omitted from P_x is perhaps that part of it which goes in commission, and that only on the supposition that no allowance is made to the agent for the sudden cutting off of his fountain of supplies. We are here touching on a point that may some day cause trouble in the actual working, both of this scheme, and of that we first investigated.

We shall now proceed to make some comparison between the results of applying profit in either of the above mentioned methods, and those that occur by using a more ordinary system; and for this purpose we shall take the most common of all, viz., that in which the cash profits allocated are applied to purchase reversionary sums. For the sake of brevity we shall consider (ε) as the symbol introduced when either of the two schemes that form the subject of this paper is being treated of, and (η) as that employed when the applying of profits to purchase reversionary sums is meant.

Then let us imagine two persons of the age x, effecting policies of the same amount, the premiums in both cases being paid annually; and in each case at ages $x+y$, $x+2y$, &c., let cash profits, ${}^1\mathbf{b}$, ${}^2\mathbf{b}$, &c., be allotted to the policies. Suppose further, that the one (ε) applies these profits, as they are allotted, to limit either the term of his policy or the number of premiums payable under it, and the other (η) buys reversionary sums, ${}^1\mathrm{B}$, ${}^2\mathrm{B}$, &c., with his, how do the two policies compare at, say age $(x+t)$? Then up to the moment when the first bonus is allocated, they are exactly the same, but ever afterwards the value of policy (ε) is greater than that of (η),—a point which will tell strongly in favour of (ε), when we consider the practical working of the various methods of distributing profit. The reason of the difference in the values is evident. For (ε), by using his bonuses to purchase endowments, exposed the *m*, (*i.e.* the bonuses) to a risk which (η) does not, and, of course, in calculating the value of his policy, he will be entitled to the worth of that risk.* Thus, throwing it into symbols, if t be not greater than n, the term at the end of which the policy becomes payable, or the premiums are to cease (ε). If $\mathrm{V}_{x|t}$= the value of the original policy at present age, and if $(x+\theta y)$ be the age at which the bonus immediately preceding age $(x+t)$ is declared—then

$$\begin{aligned}\varepsilon-\eta=&\left\{\mathrm{V}_{x|t}+{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+n}}\frac{\mathrm{D}_{x+n}}{\mathrm{D}_{x+t}}+{}^2\mathbf{b}\frac{\mathrm{D}_{x+2y}}{\mathrm{D}_{x+n}}\frac{\mathrm{D}_{x+n}}{\mathrm{D}_{x+t}}+\ldots{}^{\theta}\mathbf{b}\frac{\mathrm{D}_{x+\theta y}}{\mathrm{D}_{x+n}}\frac{\mathrm{D}_{x+n}}{\mathrm{D}_{x+t}}\right\}\\&-\left\{\mathrm{V}_{x|t}+{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}+{}^2\mathbf{b}\frac{\mathrm{D}_{x+2y}}{\mathrm{M}_{x+2y}}\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}+\ldots{}^{\theta}\mathbf{b}\frac{\mathrm{D}_{x+\theta y}}{\mathrm{M}_{x+\theta y}}\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}\right\}\end{aligned}$$

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* It will perhaps make this clearer to say that (η) exposes the Office to an additional *current* risk,—that of having to pay the reversionary bonus—but (ε) exposes the Office to no such risk; and the values of the bonuses at the outset being equal, the outstanding liability of the Office, after the expiration of any assigned interval, is therefore greater in the case of (ε) than of (η).—ED. *J. I. A.*

is the difference between the values of (ϵ) and (η),

$$= \frac{1}{D_{x+t}}\left\{({}^1\mathbf{b}D_{x+y}+{}^2\mathbf{b}D_{x+2y}+\ldots{}^\theta\mathbf{b}D_{x+\theta y}) - M_{x+t}\left({}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}+{}^2\mathbf{b}\frac{D_{x+2y}}{M_{x+2y}}+\ldots{}^\theta\mathbf{b}\frac{D_{x+\theta y}}{M_{x+\theta y}}\right)\right\}$$

But since (by hypothesis) $t > \theta y$, $\therefore M_{x+t} < M_{x+\theta y}$, and, *a fortiori*, $M_{x+t} < M_{x+y}$, $< M_{x+2y}$, &c.; and therefore (ϵ) is greater than (η): that is, the values of policies are greater when the bonuses are applied in either of the modes under consideration, than when reversionary sums are purchased with them.

The extent of the excess can be found by reducing our expression further. Thus it becomes

$${}^1\mathbf{b}\left(1-\frac{M_{x+t}}{M_{x+y}}\right)\frac{D_{x+y}}{D_{x+t}}+{}^2\mathbf{b}\left(1-\frac{M_{x+t}}{M_{x+2y}}\right)\frac{D_{x+2y}}{D_{x+t}}+\ldots{}^\theta\mathbf{b}\left(1-\frac{M_{x+t}}{M_{x+\theta y}}\right)\frac{D_{x+\theta y}}{D_{x+t}},$$

or

$${}^1\mathbf{b}\frac{D_{x+y}}{M_{x+y}}\left(\frac{M_{x+y}-M_{x+t}}{D_{x+y}}\right)\frac{D_{x+y}}{D_{x+t}}+{}^2\mathbf{b}\frac{D_{x+2y}}{M_{x+2y}}\left(\frac{M_{x+2y}-M_{x+t}}{D_{x+2y}}\right)\frac{D_{x+2y}}{D_{x+t}}+\ldots$$
$${}^\theta\mathbf{b}\frac{D_{x+\theta y}}{M_{x+\theta y}}\left(\frac{M_{x+\theta y}-M_{x+t}}{D_{x+\theta y}}\right)\frac{D_{x+\theta y}}{D_{x+t}}.$$

That is, (ϵ) exceeds (η) by the assurance of the reversionary sum that **b** can purchase at each allocation of profit, for the period from that allocation till the date of enquiry, improved at interest and mortality rates for that period.

We have said that t must not exceed n; but we only did so because if the first scheme be adopted, the policy at age $(x+n)$ terminates, and the comparison ceases. If, however, the second form be adopted, as the payment of premiums only ceases, there is no reason why t should be limited to any age under the difference between $(x+y)$ and the oldest age in the tables. The proof just given, as it was quite independent of the value of t, will suffice to embrace this case also.

Therefore whether (ϵ) stands for bonus applied to limiting the term of an assurance, or the number of payments made under it, from the time of the first declaration of profit, till the time the policy runs off the books of the Company, the value of an assurance under scheme (ϵ), will exceed that of one under similar conditions, but of which the profits are applied to purchase reversionary sums (η), by a quantity for which we have just found the expression.

We have compared (ϵ) and (η) with regard to the values, and we shall now endeavour to resolve the question, whether, if the rates of mortality and interest turn out differently from those founded on, the profit, if any, will be greater or less on system (ε), or on system (η); and here, in order to distinguish between expected and actual rates, we shall put an accent over the symbols referring to the latter.

Let, as before, $(x+t)$ be the age at which the examination takes place, and the other symbols have the same meaning as in the last problem; then, since, as we have already shown, the original policies under systems (ε) and (η) are still in the same state, and as the expression we wish to find is not one for the *amount* of the profit by either system, but simply a formula for its *difference*, we may reject altogether, as common to both, the profit made under the original policies, and narrow our investigations to a consideration of the effects produced on the cash bonuses allotted to the policies by systems (ε) and (η), subject to such conditions of mortality and interest as we have supposed. We shall also reject consideration of the intermediate bonus, as its cash value is all we can deal with, and it cannot be considered as applied in any way diverse in the one case from the other.

Then, under system (ε), at age $(x+t)$, since ${}^1\mathbf{b}$ allocated at age $(x+y)$, was invested in the purchase of an Endowment payable at, say, age $(x+s)$, of which the amount would thus be ${}^1\mathbf{b}\dfrac{D_{x+y}}{D_{x+s}}$; and the value of this at age $(x+2y)$ would be ${}^1\mathbf{b}\dfrac{D_{x+y}}{D_{x+s}}\dfrac{D_{x+s}}{D_{x+2y}}$; and this is again invested at age $(x+2y)$ in the purchase of an Endowment payable at, say, $(x+o)$; which is again valued and invested at $(x+3y)$; and so on; till at age $(x+t)$, the value of the Endowment is,

$${}^1\mathbf{b}\frac{D_{x+y}}{D_{x+s}}\frac{D_{x+s}}{D_{x+2y}}\frac{D_{x+2y}}{D_{x+0}}\cdots\frac{D_{x+\theta y}}{D_{x+n}}\frac{D_{x+n}}{D_{x+t}}, \text{ or } {}^1\mathbf{b}\frac{D_{x+y}}{D_{x+t}}.$$

Then the expression for the profit on (ε) will be

$${}^1\mathbf{b}\frac{D'_{x+y}}{D'_{x+t}}-{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+t}}-\left\{{}^1\mathbf{b}\frac{D'_{x+y}}{D'_{x+\theta y}}-{}^1\mathbf{b}\frac{D_{x+y}}{D_{x+\theta y}}\right\}\frac{D'_{x+\theta y}}{D'_{x+t}}.$$

This is the expression for the profit on (ε) for the period between $(x+\theta y)$ and $(x+t)$; the surplus up to age $(x+\theta y)$ being distributed.

It will be observed that we value the endowment at the rates founded on, not on those realised, so as not to anticipate the future profit on it.

Similarly, the expression for (η) will be

$$
\begin{aligned}
&{}^1\mathbf{b}\frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+t}}-{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left\{\frac{\mathrm{M}'_{x+y}-\mathrm{M}'_{x+t}}{\mathrm{D}'_{x+t}}+\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}\right\}\\
&-\left\{{}^1\mathbf{b}\frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+\theta y}}-{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left(\frac{\mathrm{M}'_{x+y}-\mathrm{M}'_{x+\theta y}}{\mathrm{D}'_{x+\theta y}}+\frac{\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\right)\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}
\end{aligned}
$$

By multiplying out, (ε) can be reduced to

$$-{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+t}}+{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}};$$

and similarly, (η) can be reduced to

$$
\begin{aligned}
&{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\frac{\mathrm{M}'_{x+t}}{\mathrm{D}'_{x+t}}-{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}-{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\frac{\mathrm{M}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\\
&\qquad+{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\frac{\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}.
\end{aligned}
$$

Dividing by ${}^1\mathbf{b}\dfrac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}$, we have the difference between (ε) and (η), or $(\varepsilon)\sim(\eta)$,

$$
\begin{aligned}
&{}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\left\{-\frac{\mathrm{M}_{x+y}}{\mathrm{D}_{x+t}}+\frac{\mathrm{M}_{x+y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\sim\left\{\frac{\mathrm{M}'_{x+t}}{\mathrm{D}'_{x+t}}-\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}-\frac{\mathrm{M}'_{x+\theta y}}{\mathrm{D}'_{x+t}}+\frac{\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]\\
&={}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\left\{\frac{\mathrm{M}'_{x+\theta y}}{\mathrm{D}'_{x+t}}-\frac{\mathrm{M}'_{x+t}}{\mathrm{D}'_{x+t}}\right\}\sim\left\{\frac{\mathrm{M}_{x+y}}{\mathrm{D}_{x+t}}-\frac{\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}-\frac{\mathrm{M}_{x+y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}+\frac{\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]\\
&={}^1\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\left\{\frac{\mathrm{M}'_{x+\theta y}-\mathrm{M}'_{x+t}}{\mathrm{D}'_{x+t}}\right\}\sim\left\{\frac{\mathrm{M}_{x+y}-\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}}-\frac{\mathrm{M}_{x+y}-\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]\\
&=\frac{{}^1\mathbf{b}}{\mathrm{A}_{x+y}}\left[\mathrm{A}'_{\substack{x+\theta y\\ \overline{t-\theta y}|}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\sim\left\{\mathrm{A}_{\substack{x+y\\ \overline{t-y}|}}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+t}}-\mathrm{A}_{\substack{x+y\\ \overline{(\theta-1)y}|}}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]
\end{aligned}
$$

and making the requisite changes for ${}^2\mathbf{b}$, ${}^3\mathbf{b}$ . . . ${}^\theta\mathbf{b}$, we have the expression for the total difference of profit between (ε) and (η),

$$
(8)\left\{
\begin{aligned}
&\frac{{}^1\mathbf{b}}{\mathrm{A}_{x+y}}\left[\mathrm{A}'_{\substack{x+\theta y\\ \overline{t-\theta y}|}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\sim\left\{\mathrm{A}_{\substack{x+y\\ \overline{t-y}|}}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+t}}-\mathrm{A}_{\substack{x+y\\ \overline{(\theta-1)y}|}}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]\\
&+\frac{{}^2\mathbf{b}}{\mathrm{A}_{x+2y}}\left[\mathrm{A}'_{\substack{x+\theta y\\ \overline{t-\theta y}|}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\sim\left\{\mathrm{A}_{\substack{x+2y\\ \overline{t-2y}|}}\frac{\mathrm{D}_{x+2y}}{\mathrm{D}_{x+t}}-\mathrm{A}_{\substack{x+2y\\ \overline{(\theta-2)y}|}}\frac{\mathrm{D}_{x+2y}}{\mathrm{D}_{x+\theta y}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]\\
&+\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\\
&+\frac{{}^\theta\mathbf{b}}{\mathrm{A}_{x+\theta y}}\left[\mathrm{A}'_{\substack{x+\theta y\\ \overline{t-\theta y}|}}\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\sim\left\{\mathrm{A}_{\substack{x+\theta y\\ \overline{t-\theta y}|}}\frac{\mathrm{D}_{x+\theta y}}{\mathrm{D}_{x+t}}\right\}\right]
\end{aligned}
\right.
$$

Should $^1\mathbf{b}, {}^2\mathbf{b}, \ldots {}^\theta\mathbf{b}$, be constantly increasing, so that the reversionary bonus B, is always the same, then this expression becomes

$$(9) \qquad \mathrm{B}\left[\frac{\theta(\mathrm{M}'_{x+\theta y}-\mathrm{M}'_{x+t})}{\mathrm{D}'_{x+t}} \sim \left\{\frac{\mathrm{M}_{x+y}+\mathrm{M}_{x+2y}+\ \ldots \mathrm{M}_{x+\theta y}-\theta\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}} - \frac{\mathrm{M}_{x+y}+\ \ldots \mathrm{M}_{x+(\theta-1)y}-(\theta-1)\mathrm{M}_{x+\theta y}}{\mathrm{D}_{x+\theta y}}\cdot\frac{\mathrm{D}'_{x+\theta y}}{\mathrm{D}'_{x+t}}\right\}\right]$$

and should the division of profit be annual, that is, if $y=1$, this last expression becomes

$$(10) \qquad \mathrm{B}\left[\frac{\theta(\mathrm{M}'_{x+\theta}-\mathrm{M}'_{x+t})}{\mathrm{D}'_{x+t}} \sim \left\{\frac{\mathrm{R}_{x+1}-\mathrm{R}_{x+\theta+1}-\theta\mathrm{M}_{x+t}}{\mathrm{D}_{x+t}} - \frac{\mathrm{R}_{x+1}-\mathrm{R}_{x+\theta}-(\theta-1)\mathrm{M}_{x+\theta}}{\mathrm{D}_{x+\theta}}\cdot\frac{\mathrm{D}'_{x+\theta}}{\mathrm{D}'_{x+t}}\right\}\right]$$

Having now obtained $(\varepsilon)\sim(\eta)$, let us try to discover whether (ε) or (η) is the greater; and for simplicity, let us assume $t=2y$; that is, a bonus of $^1\mathbf{b}$ being declared on each policy at age $(x+y)$, with which in the one case (ε) an endowment payable at $(x+n)$ is bought, and in the other (η) a reversionary sum; what is the difference of the profit in the two cases, at age $(x+2y)$?

Making the necessary substitutions in (1), we have

$$^1\mathbf{b}\,\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\mathrm{A}'_{\substack{x+y\\ \overline{y}|}}\,\frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}} \sim \mathrm{A}_{\substack{x+y\\ \overline{y}|}}\,\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}}\right]$$

—an expression symmetrical enough.

If the rates of mortality and interest actually realised, be precisely the same as those founded on, then the dashes will disappear: (ε) will equal (η), and $(\varepsilon)\sim(\eta)$ become nil;—thus proving that an office acting under system (ε), loses as much by the premature death of a policyholder, as an office acting on system (η), even though it pays less. We should hardly call attention to this, were it not a popular argument in support of such schemes of dividing profit as those now treated of. It needed no formula, however, to show this, as it will be at once seen, that whatever benefit was gained by the less payments under the death claims, was discounted in the sale of the endowments. In fact, the same thing was shown, when it was proved that the values of such policies were greater than by methods when the assured grasped at less and ran less risk.

When the experienced mortality and interest are different from the expected, (*i.e.* those by which the benefits are sold), (ε) does not necessarily equal (η).

We shall first assume the experienced and the anticipated interest the same, but the actual mortality more favourable than the expected. Then, in the expression,

$$\overset{1}{}\mathbf{b}\frac{\mathrm{D}_{x+y}}{\mathrm{M}_{x+y}}\left[\overset{(\varepsilon)}{\mathrm{A}'_{\substack{x+y\\ \overline{y}\rceil}}\frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}}}\sim\overset{(\eta)}{\mathrm{A}_{\substack{x+y\\ \overline{y}\rceil}}\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}}}\right]$$

$$\mathrm{A}'_{\substack{x+y\\ \overline{y}\rceil}}<\mathrm{A}_{\substack{x+y\\ \overline{y}\rceil}} \text{ and } \frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}}<\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}}$$

and therefore (ε) is less than (η).

This is also evident, when we remember that a mortality lighter than anticipated will cause the endowments to be sold at a loss, but the reversionary sums at a profit.

If, on the other hand, the mortality be heavier than provided for by the tables, (ε) will exceed (η), and the endowments will be sold at a profit, and the reversions at a loss.

Next, should we assume the mortality rates, actual and anticipated, the same, but the interest realised greater than that assumed, then,

$$\mathrm{A}'_{\substack{x+y\\ \overline{y}\rceil}}<\mathrm{A}_{\substack{x+y\\ \overline{y}\rceil}} \text{ but } \frac{\mathrm{D}'_{x+y}}{\mathrm{D}'_{x+2y}}>\frac{\mathrm{D}_{x+y}}{\mathrm{D}_{x+2y}};$$

so that in order to find whether $(\varepsilon)-(\eta)$ is a positive quantity, we shall have to analyse the expression still further.

Leaving out the common multiplier (a positive factor), $(\varepsilon)-(\eta)$ is

$$\frac{(l_{x+y}-l_{x+y+1})v'+(l_{x+y+1}-l_{x+y+2})v'^2+\ldots(l_{x+2y-1}-l_{x+2y})v'^y}{l_{x+y}}\cdot\frac{l_{x+y}}{l_{x+2y}v'^y}$$

$$-\frac{(l_{x+y}-l_{x+y+1})v+(l_{x+y+1}-l_{x+y+2})v^2+\ldots(l_{x+2y-1}-l_{x+2y})v^y}{l_{x+y}}\cdot\frac{l_{x+y}}{l_{x+2y}v^y}$$

Divide by $\frac{l_{x+y}}{l_{x+2y}}$, and multiply by $v'^y v^y$, (positive quantities, and therefore having no effect on the sign of the expression), and we have

$$v^y\{(l_{x+y}-l_{x+y+1})v'+\ldots(l_{x+2y-1}-l_{x+2y})v'^y\}$$
$$-v'^y\{(l_{x+y}-l_{x+y+1})v+\ldots(l_{x+2y-1}-l_{x+2y})v^y\}$$

which is the same as

$$(v^y v'-v'^y v)(l_{x+y}-l_{x+y+1})+(v^y v'^2-v'^y v^2)(l_{x+y+1}-l_{x+y+2})\ldots$$
$$+(v^y v'^y-v'^y v^y)(l_{x+2y-1}-l_{x+2y}).$$

But each of these terms is positive, since dividing each term respectively by $vv'(l_{x+y}-l_{x+y+1})$, $v^2v'^2(l_{x+y+1}-l_{x+y+2})$, ... $v^yv'^y(l_{x+2y-1}-l_{x+2y})$ (all positive quantities) we have quotients

$$v^{y-1}-v'^{y-1},\ v^{y-2}-v'^{y-2},\ \ldots.,\ 1-1.$$

Each, since v is greater than v', a positive term, except the last, which is zero, and therefore the whole expression is positive, that is $(\epsilon)-(\eta)$ is positive, and therefore, looking to interest alone, if a greater rate of interest be realised than provided for in the tables founded on, there will be a greater profit on policies under either of the two schemes this paper treats of, than under the more ordinary method of declaring the bonus in the form of a reversionary sum.

If both the interest and mortality experienced be more favourable than anticipated, there will be under system (ϵ), considering both *original policy and bonuses*, less profit on the mortality, and greater on the interest, than under system (η).

Whether, in that case, (ϵ) or (η) will be more profitable to the Assurance Company, can of course only be determined in each particular case by applying the values given to formula (8).

We have considered the question with a view only to the schemes we have been treating of; but, (8), the formula arrived at will serve equally well to determine whether an Endowment, or an assurance, for which equal cash sums have been paid as premiums, is the more profitable to a Company, and will also give the amount of the difference of profit.

It may be well to illustrate this part of the paper by an actual example.

Two persons, each aged, say 30, take out policies of £100 payable at death in an Assurance Company. Each is charged the same annual premium; and after 5 years, the same cash bonus of, say ·651, is declared on each policy, the one sinks his bonus in the purchase of an Endowment payable at, say age 75, the other buys a reversionary sum (£1. 10*s.*) with his. The office sells both endowment and reversion at Carlisle 3 per cent. The realised rates are $\frac{9}{10}$ Carlisle 4 per cent. On which policy is there greater profit at, say, age 40, and what is the then difference between the two?

The formula for $(\epsilon\sim\eta)$ is here

$$1\cdot5\left(A'_{35\,\overline{5}|}\frac{D'_{35}}{D'_{40}}\sim A_{35\,\overline{5}|}\frac{D_{35}}{D_{40}}\right)$$

where the first quantity within the bracket (ϵ) is calculated at $\frac{9}{10}$ Carlisle 4 per cent, and the second (η) at Carlisle 3 per cent. When we work this out, we shall find that (η) exceeds (ϵ) in the given time by ·00768, and that, thus, the greater profit on interest realised by (ϵ), is more than counterbalanced by the loss on the sale of the endowments, through the lighter mortality.

As an illustration of our analysis concerning interest and mortality, considered separately, let us take this question; and assuming, first, that the interest anticipated and realised is the same (3 per cent), but that the mortality experience turns out in conformity with the $\frac{9}{10}$ Carlisle instead of the Carlisle table, as founded on: and, secondly, that the Carlisle table represents the mortality, both actual and anticipated, but the interest made be 4 per cent, instead of 3 per cent as founded on—we shall find that, in the first case, (η) will exceed (ϵ) by ·00927, and in the second, (ϵ) will exceed (η) by ·00177.

We shall very briefly glance at another interesting question.

In examining the actual mortality that occurred among policyholders under scheme (ϵ) or scheme (η), we have considered that, $(x+t)$ being any age, $\frac{l'_{x+t}}{l'_x}$ will, by the experience table, correctly represent *under either system* the probability of a life aged x living to age $(x+t.)$ No doubt this would be correct, were there no disturbing circumstances; but this is not always the case. Several Offices give their assurers the option of choosing between scheme (ϵ) and scheme (η), at the time the first bonus is declared on the policy; and some, if we correctly understand the wording of their prospectuses, even permit this to be done at each declaration of profit. In such cases as these, it is evident that though $\frac{l'_{x+t}}{l'_x}$ may represent the probability, taken from the experience of all the lives together, of a life aged x living t years; yet it will be a very different thing when the sickly man is permitted with his cash bonus to buy a reversion, and the robust man an endowment. If it be only done once, and a person having chosen his system be confined to it, the effect of the selection of schemes by the assurer may perhaps die out before long; but it is quite another matter when this disturbing cause becomes perennial by the policyholder making his election at each declaration of profit. With all deference to the opinion of the actuarial advisers of those Offices that allow this latter thing to be done, we cannot but think it an unfortunate permission.

In the first place let us see what is actually allowed. If the bonus be declared, as it usually is, in the form of a reversionary sum; since, by the operation of applying bonus to limiting the term of the assurance, there will be an unusually large number of Endowment Assurances on the books of the Company, we shall have to choose at each declaration between increasing the benefit to the assurers who have at any previous time adopted this system, at the expense of those who have confined themselves to system (η); or by declaring a less reversionary bonus in the one case than the other, disgust those policyholders who no longer wish to pursue system (ϵ). This, we admit, is more a popular than a theoretical objection; but still we think it is none the less forcible, and it is better to avoid the possibility of its being raised.

Again, what are we to understand by the statement that a person may have the choice of either method at each time of division of profit? Does it mean that at each investigation period a person has the power of dealing only with the reversionary bonus then declared, or with all of the others as well? Very possibly these Offices might answer, he can only apply the bonus then declared, to hasten the term of payment of the sum assured, or the extinction of his premiums; but this (though not so bad as allowing him to deal with all bonuses) involves the giving the persons who have already adopted system (ϵ), an advantage over those who, having previously adopted (η), now wish to adopt (ϵ); as, in the former case, from the nature of the formula, all the previous bonuses are dealt with, as well as the one then declared—being surrendered at full value, and sunk in the purchase of a new endowment of shorter term; a thing that a Company would hardly allow to a policyholder, who came and frankly asked for the surrender value of his endowment policy, because, owing to failing health, he could hardly expect to live till it matured, and would therefore prefer to buy, with the sum paid him by the Office, a reversion, or at least an Endowment, payable at a time he would have more chance of reaching.

No doubt, at present, under system (η), an assurer, whatever be his state of health, is allowed to surrender his reversionary bonuses for cash; or, *vice versâ*, with his cash profit, purchase a reversion; but this, an evil in itself, though it has in certain cases its advantages, is much aggravated, when this cash sum is allowed to fructify as an endowment, which at periodical intervals the policyholder has the power of reconsidering, with a view, if need be, to altering it to what may best suit his interests.

In fact, giving this option to an assurer, is practically the same thing as if an Office were to allow its policyholders to surrender their policies at the full amount reserved against them, a thing which, so far as I am aware, no Company would dream of doing; and even should, as in the case of policy surrenders, a deduction be made from the full value, this deduction will be, as in that case, purely empirical.

If, then, either of the schemes that are the subject of this paper, and more especially the first of the two, be chosen, it will be much better to confine a person who has once adopted it, to it, and at each investigation only permit him to apply his bonus to shorten the term of the Endowment.

A comparison arises at once in the mind of everyone between policies on either of these systems, and those corresponding, in which n is fixed at once. Which has the best of it?

Should both assurers buy at cost price, whether that cost price be different from the rates founded on, or not, the time of payment in each case will be the same;* but as the first has to buy at

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* A proof of this is hardly necessary; perhaps, however, it may be well to furnish one.

Let two assurers with any Office each pay the premium P_x—the one, purchasing an Endowment Assurance payable at $(x+\nu)$, without additions—the other, an ordinary assurance payable at death, with additions, the amount of this assurance being assumed the same. The latter invests these additions in endowments which became finally payable at age $(x+n)$. Which is least, n or ν?

The premium P_x each pays, is made up in the following way. There is a premium ϖ_x, which being invested at realised rates, will secure the sum contracted for, *at death*. This, together with the addition for expenses, and the margin from which springs the profit, makes up P_x. This margin we shall call f.

In the Endowment Assurance *uncertain*, (we shall take for illustration the *first* of the two systems merely), we have seen the formula is, when all the rates are the same, *i.e.* the realized rates,

$$N_{x+n-1}=(1\times a_x)({}^1\mathbf{b}D_{x+y}+{}^2\mathbf{b}D_{x+2y}+\ldots\ldots{}^\theta\mathbf{b}D_{x+\theta y}).$$

It must be remembered, as we there said, that these formulæ do not allow the policyholder any profit for the period from $(x+\theta y)$ to $(x+n)$. Let ${}^i\mathbf{b}$ be this profit, and as we suppose the policyholder to reap now all benefit possible, we must add ${}^i\mathbf{b}$ to our formula, which will thus become

$$N_{x+n-1}=(1+a_x)({}^1\mathbf{b}D_{x+y}+{}^2\mathbf{b}D_{x+2y}+\ldots\ldots{}^\theta\mathbf{b}D_{x+\theta y}+{}^i\mathbf{b}D_{x+n}).$$

Now as f is the only source of profit, we can calculate the value of $\mathbf{b}$. Thus

$${}^1\mathbf{b}=f\frac{N_{x-1}-N_{x+y-1}}{D_{x+y}},\ {}^2\mathbf{b}=\frac{N_{x+y-1}-N_{x+2y-1}}{D_{x+2y}},\ {}^i\mathbf{b}=\frac{N_{x+\theta y-1}-N_{x+n-1}}{D_{x+n}};$$

and substituting these values of $\mathbf{b}$ in the formula, we have

$$N_{x+n-1}=f(N_{x-1}-N_{x+n-1})(1+a_x).$$

Again, in the Endowment assurance certain, since $(x+\nu)$ is the age at which the sum assured is payable, then, at that age, for ϖ_x (the premium actually necessary to provide assurance at death) the policyholder will have enjoyed a temporary assurance of £1, and an endowment, then payable, of $1-\dfrac{1+a_{x+\nu}}{1+a_x}$ (that is, the surrender value of a whole life something over prime cost, and receives no bonus, while the second has *his* overpayment returned in this form to him, *theoretically, n* in the latter case should be of earlier date. *Practically,* though we are aware that it involves the somewhat paradoxical statement that a "with-profit" policyholder reaps little or no profit, we do not think there will be any great difference. This can of course only be tested by actual example. Thus, the with-profit premium at age 25 for assurance of £100 at death, we shall assume to be 2·1286, (Carlisle 3 per cent with 25 per cent added), and the bonus declared to be equal to a reversionary sum of £1. 10*s*. per cent per annum. This is, we think, at the very least, not under the average profit that Offices charging such a premium would declare. (The annual premium for such a bonus at Carlisle 4 per cent is ·59154.)

We shall assume the first scheme to be adopted; and this bonus, declared quinquennially, to be surrendered at Carlisle 4 per cent, and the Endowments purchased, and the policy valued, at Carlisle 3 per cent. The following will then show the effect of this system:—

Ages marking beginning of different statuses of Policy.		Time when Policy will be payable.	
25		At death	
30		75	At 1st investigation
35		71	„ 2nd „
40		68	„ 3rd „
45		66	„ 4th „
50		64	„ 5th „
55		63	„ 6th „
60		62	„ 7th „

So that the assurance becomes payable at age 62 finally.

We have before us the prospectus of at least one Company of the highest standing, that would grant to a person of the age of

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assurance); but he then receives £1, so (as $\varpi_x + f$ is the premium, less expenses, he actually pays), f must be the premium for an endowment of $1 - \left(1 - \frac{1 + a_{x+\nu}}{1 + a_x}\right)$. Whence $f = \frac{N_{x+\nu-1}}{(N_{x-1} - N_{x+\nu-1})} \frac{D_x}{N_{x-1}}$. Substituting this for f, in the expression given above, we have,

$$N_{x+n-1} = \frac{N_{x+\nu-1}(N_{x-1} - N_{x+n-1})}{N_{x-1} - N_{x+\nu-1}},$$

$$\therefore \frac{N_{x+n-1}}{N_{x-1} - N_{x+n-1}} = \frac{N_{x+\nu-1}}{N_{x-1} - N_{x+\nu-1}}$$

Whence, from the construction of Commutation Tables, N_{x+n-1} must be equal to $N_{x+\nu-1}$, and therefore $n = \nu$.—Q.E.D.

25, for such a premium as the one given above, an endowment assurance payable at 64; and those of others whose rates at 65 closely approximate to this premium, so that the benefit obtainable by an assurer on this principle, should even these highly favourable rates of bonus, surrender values of policy, &c., be allowed in the calculation, does not exceed two or three years.

In the example given above, though we might have at once proceeded to the final age at which the policy will become payable, without finding the intermediate ages, yet we have preferred to give the intervening ages also, as the series will elucidate a peculiarity of these systems of applying bonus, that well deserves to be pointed out. It will be noticed that the same reversionary bonus which, towards the beginning of the series, will bring the term of payment nearer by four years, will, at the end, only bring it nearer by one year. Had we chosen a really equal bonus at each division of surplus, instead of an increasing one, as a constant reversionary bonus necessarily is, this peculiarity would have been even more striking. Of course the reason is to be found in the great cheapness at which endowments for long terms can be purchased, and their much heavier cost when the probability of their being enjoyed is great, as it necessarily becomes when the period of their maturity is close at hand.

We have now, we think, gone over most of the important features that these systems of dividing profit present. They are methods which we consider are likely to grow in favour with the public; and whatever opinion actuaries may hold as to the adviseableness of their introduction, it will be generally admitted that they are extremely interesting methods of dividing the profits realised by an Assurance Company.

On the Distribution of Profits in Mutual Insurance Societies. By Professor Pell, *of the Sydney University.*[*]

[Read at the meeting of the Philosophical Society, on Wednesday, 7th December, and reprinted from the *Sydney Morning Herald*, 10th December, 1864.]

THERE is no part of the subject of Life Insurance which has occasioned so much difficulty, and given rise to so much diversity of opinion and of practice, as that of the distribution of profits.

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[*] We reprint with pleasure this paper, for which we are indebted to the kindness of a correspondent. The arguments contained in it appear to us very ingenious and forcibly put, although far from convincing. It would require more time than is at present at our command to give a full examination of Professor Pell's arguments, but we have briefly indicated the points in which they appear to be defective.—Ed. *J. I. A.*